Kinematics and Inverse Kinematics for the Humanoid Robot HUBO2+

Rowland O'Flaherty†, Peter Vieira†, Michael Grey†,
Paul Oh‡, Aaron Bobick†, Magnus Egerstedt†, and Mike Stilman†

Abstract—**This paper derives the forward and inverse kinematics of a humanoid robot. The specific humanoid that the derivation is for is a robot with 27 degrees of freedom but the procedure can be easily applied to other similar humanoid platforms. First, the forward and inverse kinematics are derived for the arms and legs. Then, the kinematics for the torso and the head are solved. Finally, the forward and inverse kinematic solutions for the whole body are derived using the kinematics of arms, legs, torso, and head.**

I. Introduction

In this paper we present a derivation of the forward kinematics (FK) and inverse kinematics (IK) of a humanoid robot with 27 degrees of freedom, specifically the HUBO 2+ platform (Hubo). A picture of Hubo is shown in Fig. 1. The FK and IK are not solved for the entire 27 joints but instead broken up into six parts. These parts consist of the two arms (six joints each), the two legs (six joints each), the torso (1 joint), and the head (2 joints). The majority of the paper is spent on the arms and the legs because the head and torso are almost trivial. However, even though the head and torso are simple to solve for as individual parts they can not be ignored because for a full body FK and IK they are essential components.

Having the forward kinematics (FK) and inverse kinematics (IK) is crucial in any robot manipulator especially for a humanoid robot. This is because usually it is desirable to control the end-effector of the robot (e.g. the hand of one arm) in its workspace, not in joint space. In other words, tell the hand of the humanoid to go to point (x, y, z) not tell the joints to go to values θ_i. However, the joint values are what can be directly controlled, therefore, one needs to know how to convert from workspace to joint space and vice versa.

Ali et. al. presented a closed-form solution for the inverse kinematics (IK) of the limbs of the HUBO2+ robot platform [1]. They used a reverse decoupling mechanism method by viewing the kinematic chain of a limb in reverse order and decoupling the position and orientation. The authors then used the inverse transform method to compute eight possible solutions for each limb. The correct solution is selected based on joint limits and constraints. In working through their solution, discrepancies were found in the calculations. We corrected the errors and solved for the IK of all four limbs for our HUBO2+ humanoid robot. Using the IK for the four limbs and the head and torso we have formulated an IK for the entire robot.

This work was supported in part by DARPA #N65236-12-1-1005: DARPA Robotics Challenge and NSF CNS-0960061 MRI-R2: Unifying Humanoids Research.

†Authors are affiliated with Georgia Institute of Technology, Atlanta, GA, 30332, USA. {rowland.oflaherty, plvieira, mxgrey, magnus}@gatech.edu and {afb,mstilman}@cc.gatech.edu.

‡Author is affiliated with Drexel Univeristy, Philadelphia, PA, 19104, USA. paul@coe.drexel.edu.

Fig. 1. The Humanoid Robot Hubo2+ for which we calculate kinematics.

II. Hubo Geometry and Kinematics

In order to control a humanoid using workspace control both forward and inverse kinematic solutions are required. The solution to this problem involves solving for the joint angles given a desired position and orientation while accounting for singularities, joint limits and feasible workspace issues.

The kinematic structure of the right and left side of Hubo are identical, therefore, the left and right arms have the same joint coordinate frames and Denavit-Hartenberg (DH) parameters, as do the left and right legs. The only difference between the left and the right limbs is the offset direction from the base frame. We first go through the solution for the IK of the arms, and then the legs. The joint coordinate frames are shown in Fig. 2 and the length of each link is shown in Table I. The DH parameters (using the standard convention) for the arms and legs are shown in Table II and Table III, respectively.

A. Forward Kinematic Solution for the Arm

The forward kinematics problem is that of solving for the end-effector orientation and position given the joint angles.

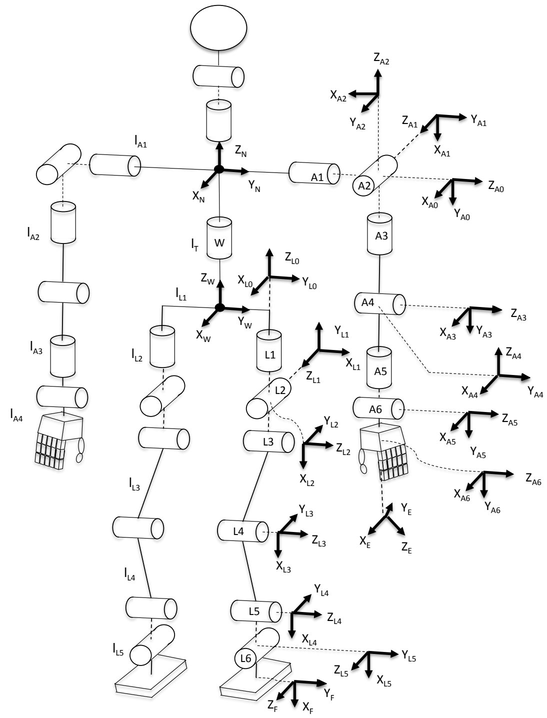

Fig. 2. Hubo coordinate frames

TABLE I. LINK LENGTHS OF HUBO

Lengths of arm links	
Link	Length (mm)
l_{A1}	215
l_{A2}	179
l_{A3}	182
l_{A4}	121
l_E	178

Lengths of leg links	
Link	Length (mm)
l_T	187
l_{L1}	88
l_{L2}	182
l_{L3}	300
l_{L4}	300
l_{L5}	95

This is easily solved using the robot geometry and coordinate frames, which are specified in the DH parameters. The general homogeneous transformation from one link to the next given the DH parameters is represented in matrix form as:

$$^{i-1}T_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & \mathrm{a}_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & \mathrm{a}_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & \mathrm{d}_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (1)$$

where $^{i-1}T_i$ is the transformation from coordinate frame $i-1$ to frame i. The base frame for the arm is at the neck, and its tranformation to the first shoulder joint is

$$^{N}T_{A0} = \begin{bmatrix} 0 & 0 & 1 & l_{A1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

An additional transformation $^{6}T_E$ is used for the transformation from the hand to the end-effector. In order calculate the forward kinematics (FK), the six transformation matrices from each joint are pre-multiplied to obtain the position and orientation of the end-effector relavtive to the shoulder. We define the transformation from the shoulder to the hand as

$$^{A0}T_{A6} = \,^{A0}T_{A1}\,^{A1}T_{A2}\,^{A2}T_{A3}\,^{A3}T_{A4}\,^{A4}T_{A5}\,^{A5}T_{A6}. \quad (3)$$

Thus, when solving for the forward kinematics of the end-effector with respect to the neck $^{A0}T_{A6}$ must be pre-multiplied by $^{N}T_{A0}$ and post-multiplied by $^{A6}T_E$. Therefore, given a set of joint angles the FK is calculated as

$$^{N}T_E = \,^{N}T_{A0}\,^{A0}T_{A6}\,^{A6}T_E. \quad (4)$$

TABLE II. DH PARAMETERS OF THE ARMS

Right arm DH parameters				
Coord. Frame i	θ_i	α_i	a_i	d_i
1	$\theta_1 + \pi/2$	$\pi/2$	0	0
2	$\theta_2 - \pi/2$	$\pi/2$	0	0
3	$\theta_3 + \pi/2$	$-\pi/2$	0	$-l_{A2}$
4	θ_4	$\pi/2$	0	0
5	θ_5	$-\pi/2$	0	$-l_{A3}$
6	$\theta_6 + \pi/2$	0	l_{A4}	0

B. Inverse Kinematic Solution for the Arm

The inverse kinematics is the problem of solving for the joint angles given the end-effector orientation and position, specified as $^{N}T_E$. This is a much harder problem because there are multiple solutions. When solving the inverse kinematics of a manipulator, Pieper [2] indicates that a closed-form solution exists if three consecutive joint axes of the manipulator are parallel to one another, or intersect at a single point. The three shoulder joint axes on the Hubo intersect at a single point for the arms, and the three hip joints intersect at a single point for the legs, therefore a closed-form kinematic solution exists for both the arms and the legs.

We will solve the IK problem from the shoulder to the hand by using the transformation $^{A0}T_{A6}$. This is obtained by pre-multiplying $^{N}T_E$ by $^{A0}T_N$ and post-multiplying by $^{E}T_{A6}$. Thus,

$$^{A0}T_{A6} =^{A0} T_N\,^{N}T_E\,^{E}T_{A6}. \quad (5)$$

Let us write $^{A0}T_{A6}$ obtained from $^{N}T_E$ as

$$^{A0}T_{A6} = \begin{bmatrix} \mathrm{x}_6 & \mathrm{y}_6 & \mathrm{z}_6 & \mathrm{p}_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{n} & \mathrm{s} & \mathrm{a} & \mathrm{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (6)$$

where x_6, y_6, and z_6 are the unit vectors along the principal axes of the hand frame and p_6 is the position vector describing the location of the hand relative to the shoulder. These three unit vectors describe the orientation of the hand coordinate frame relative to the shoulder coordinate frame. The vectors n, s, a, and p represent the normal vector, sliding vector, approach vector, and position vector of the hand, respectively [3].

Using this knowledge, the arm can be viewed in reverse so that the last three joints make up the shoulder, thus the position and orientation of the shoulder frame can be described relative to the hand frame. This new position vector, p', is only a function of θ_4, θ_5 and θ_6, and thus decouples the arm into position and orientation components. The IK problem is solved in this reverse method by taking the inverse of both sides of (6).

$$^{A0}T_{A6}{}' = \begin{bmatrix} \mathrm{n} & \mathrm{s} & \mathrm{a} & \mathrm{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}' = \begin{bmatrix} \mathrm{n}' & \mathrm{s}' & \mathrm{a}' & \mathrm{p}' \\ 0 & 0 & 0 & 1 \end{bmatrix} = \,^{A6}T_{A0} \quad (7)$$

As in [1], we will now show how to use this reverse method to solve for the joint angles of the right arm. To solve for the last three joint angles (i.e., the elbow, wrist yaw and wrist pitch angles) we can use the inverse transform method [1] by multiplying both sides of (7) by ${}^{A5}T_{A6}$. This results in an equation where the left side of the equation is[1],

$$G_{5-arm}^{(left)} = {}^{A5}T_{A6} \begin{bmatrix} \mathrm{n}' & \mathrm{s}' & \mathrm{a}' & \mathrm{p}' \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} -S_6 n'_x - C_6 n'_y, & -S_6 s'_x - C_6 s'_y, & -S_6 a'_x - C_6 a'_y, & -S_6(p'_x + l_{A4}) - C_6 p'_y \\ C_6 n'_x - S_6 n'_y, & C_6 s'_x - S_6 s'_y, & C_6 a'_x - S_6 a'_y, & C_6(p'_x + l_{A4}) - S_6 p'_y \\ n'_z, & s'_z, & a'_z, & p'_z \\ 0, & 0, & 0, & 1 \end{bmatrix}, \quad (8)$$

and the right side of G_5 is

$$G_{5-arm}^{(right)} = {}^{A5}T_{A0} = {}^{A5}T_{A4}\, {}^{A4}T_{A3}\, {}^{A3}T_{A2}\, {}^{A2}T_{A1}\, {}^{A1}T_{A0} =$$
$$\begin{bmatrix} g_{511} & g_{512} & g_{513} & -l_{A2} C_5 S_4 \\ g_{521} & g_{522} & g_{523} & -l_{A2} C_4 - l_{A3} \\ g_{531} & g_{532} & g_{533} & l_{A2} S_4 S_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (9)$$

We can solve for the last three joint angles by equating the position terms of (8) and (9) to get

$$S_6(p'_x + l_{A4}) + C_6 p'_y = S_4 C_5 l_{A2} \quad (10)$$
$$C_6(p'_x + l_{A4}) - S_6 p'_y = -C_4 l_{A2} - l_{A3} \quad (11)$$
$$p'_z = l_{A2} S_4 S_5. \quad (12)$$

The joint angle θ_4 is solved for by first letting $p'_x + l_{A4} = rC_\psi$ and $p'_y = rS_\psi$ and then substituting these into (10) and (11) in addition to using the trigonometric sum identities to obtain,

$$rS_{6\psi} = S_4 C_5 l_{A2} \quad (13)$$
$$rC_{6\psi} = -C_4 l_{A2} - l_{A3} \quad (14)$$
$$p'_z = S_4 S_5 l_{A2} \quad (15)$$

where $r = \sqrt{(p'_x + l_{A4})^2 + (p'_y)^2}$ and $\psi = \mathrm{atan2}(p'_y, p'_x + l_{A4})$ [2]. By then squaring (13), (14) and (15) and adding them, we can obtain an equation for C_4:

$$C_4 = \frac{(p'_x + l_{A4})^2 + {p'_y}^2 + {p'_z}^2 - l_{A2}^2 - l_{A3}^2}{2 l_{A2} l_{A3}},$$

from which we can obtain the elbow joint solution,

$$\theta_4 = \mathrm{atan2}\left(\pm \mathrm{real}\left(\sqrt{1 - C_4^2}\right), C_4\right). \quad (16)$$

From (15) we solve for S_5, and from that we solve for the wrist yaw joint solution θ_5,

$$S_5 = \frac{p'_z}{S_4 l_{A2}},$$
$$\theta_5 = \mathrm{atan2}\left(S_5, \pm \mathrm{real}\left(\sqrt{1 - S_5^2}\right)\right). \quad (17)$$

To obtain the wrist pitch joint solution θ_6 we first divide (13) by (14) to get

$$\tan(\theta_6 + \psi) = \frac{S_{6\psi}}{C_{6\psi}} = \frac{S_4 C_5 l_{A2}}{-C_4 l_{A2} - l_{A3}}, \quad (18)$$

from which we solve for θ_6 as[3]

$$\theta_6 = \mathrm{wrapToPi}(\mathrm{atan2}(S_4 C_5 l_{A2}, -C_4 l_{A2} - l_{A3}) - \psi). \quad (19)$$

To solve for the three shoulder joints we calculate G_3 by pre-multiplying (8) and (9) by ${}^{A4}T_{A5}$ and ${}^{A3}T_{A4}$ to obtain,

$$G_{3-arm}^{(left)} = {}^{A3}T_{A4}\, {}^{A4}T_{A5}\, {}^{A5}T_{A6} \begin{bmatrix} \mathrm{n}' & \mathrm{s}' & \mathrm{a}' & \mathrm{p}' \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} g_{311} & g_{312} & g_{313} & g_{314} \\ g_{321} & g_{322} & g_{323} & g_{324} \\ g_{331} & g_{332} & g_{333} & g_{334} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (20)$$

on the left hand side and

$$G_{3-arm}^{(right)} = {}^{A3}T_{A0} = {}^{A3}T_{A2}\, {}^{A2}T_{A1}\, {}^{A1}T_{A0} =$$
$$\begin{bmatrix} C_1 C_3 + S_1 S_2 S_3 & C_3 S_1 - C_1 S_2 S_3 & C_2 S_3 & 0 \\ -C_2 S_1 & C_1 C_2 & S_2 & -l_{A2} \\ C_3 S_1 S_2 - C_1 S_3 & -S_1 S_3 - C_1 C_3 S_2 & C_2 C_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (21)$$

on the right hand side. By comparing element (2,3) of (20) and (21) we obtain S_2, and use that to get C_2 and finally the shoulder roll joint solution θ_2. Thus,

$$S_2 = g_{323} = a'_x(C_4 C_6 - C_5 S_4 S_6) - a'_y(C_4 S_6 + C_5 C_6 S_4) - a'_z S_4 S_5,$$
$$\theta_2 = \mathrm{atan2}\left(S_2, \pm \mathrm{real}\left(\sqrt{1 - S_2^2}\right)\right). \quad (22)$$

We compare elements (1,3) and (3,3) of (20) and (21) to obtain two equations. If we divide these we obtain S_3 and C_3 and then solve for the shoulder yaw joint solution θ_3,

$$\frac{g_{313}}{g_{333}} = \frac{C_2 S_3}{C_2 C_3},$$

$$S_3 = g_{313} = -a'_x(C_6 S_4 + C_4 C_5 S_6) + a'_y(S_4 S_6 - C_4 C_5 C_6) - a'_z C_4 S_5,$$
$$C_3 = g_{333} = -a'_x S_5 S_6 - a'_y C_6 S_5 + a'_z C_5,$$
$$\theta_3 = \mathrm{atan2}(S_3, C_3). \quad (23)$$

We perform the same procedure to obtain the last joint solution, θ_1, which is the shoulder pitch angle. We compare elements (2,1) and (2,2) of (20) and (21) and divide them to obtain,

$$\frac{g_{321}}{g_{322}} = \frac{-C_2 S_1}{C_2 C_1},$$

$$S_1 = -n'_x(C_4 C_6 - C_5 S_4 S_6) + n'_y(C_4 S_6 + C_5 C_6 S_4) + n'_z S_4 S_5,$$
$$C_1 = s'_x(C_4 C_6 - C_5 S_4 S_6) - s'_y(C_4 S_6 + C_5 C_6 S_4) - s'_z S_4 S_5,$$
$$\theta_1 = \mathrm{atan2}(S_1, C_1). \quad (24)$$

There are two solutions for θ_2, θ_4, and θ_5, therefore there are eight total solutions to the arm IK. When the goal position is outside the feasible workspace of the limb, the joint solutions will have imaginary parts. To deal with this, we take only the

[1]The sine and cosine of an angle α is abbreviated S_α and C_α, respectively.
[2]$\mathrm{atan2}(y, x)$ is the two argument arc tangent function.
[3]$\mathrm{wrapToPi}(\alpha)$ wraps the angle α to the interval between $-\pi$ and π.

real parts, which in turn gives the solution that is closest to the desired position. Furthermore, there are five cases that result in singularities, and details of the solution methods can be viewed in [1]. The following are our final equations for each case of the singularity conditions.

1) Case 1 (elbow singularity): When $\theta_4 = 0$ joints θ_3 and θ_5 are collinear, thus an infinite number of solutions exist to orient the end-effector in the desired orientation. We set θ_3 to its previous value and define $\theta_S = \theta_5 + \theta_3$. These joint angles are solved by first solving for θ_6 by using elements (2,4) and (1,4) of (5) and (6). Then solving for θ_S by using elements (1,3) and (3,3) of (8) and (9). Finally, θ_5 is found from θ_S. Thus,

$$\begin{aligned}
&\theta_6 = \operatorname{atan2}\left(\frac{p'_y}{l_{A2}+l_{A3}}, \frac{-l_{A4}-p'_x}{l_{A2}+l_{A3}}\right),\\
&\theta'_S = \operatorname{atan2}(-C_6 a'_y - S_6 a'_x, a'_z),\\
&\theta_S = \begin{cases} \theta'_S & \text{if} \quad C_2 \geq 0 \\ \text{wrapToPi}(\theta'_S + \pi) & \text{if} \quad C_2 < 0 \end{cases},\\
&\theta_5 = \text{wrapToPi}(\theta_S - \theta_3).
\end{aligned}$$

2) Case 2 (shoulder singularity): When $\theta_2 = \pi/2$ (for the left arm) or $-\pi/2$ (for the right arm) joints θ_1 and θ_3 are collinear. The same approach as above is performed. However, $\theta_S = \theta_3 \pm \theta_5$ and solving for it uses elements (2,1) and (2,2) of (8) and (9). This results in the following

$$\begin{aligned}
&\theta'_S = \operatorname{atan2}(S_6 s'_y - C_6 s'_x, S_6 n'_y - C_6 n'_x),\\
&\theta_S = \begin{cases} \theta'_S & \text{if} \quad S_4 \geq 0 \\ \text{wrapToPi}(\theta'_S + \pi) & \text{if} \quad S_4 < 0 \end{cases},\\
&\theta_1 = \begin{cases} \text{wrapToPi}(\theta_S + \theta_3) & \text{for left arm} \\ \text{wrapToPi}(\theta_S - \theta_3) & \text{for right arm} \end{cases}.
\end{aligned}$$

3) Case 3 (elbow-shoulder singularity): When $\theta_4 = 0$ and $\theta_2 = \pi/2$ (for the left arm) or $-\pi/2$ (for the right arm) joints θ_1, θ_3 and θ_5 are collinear. The same approach as above is taken. However, $\theta_S = \theta_1 \pm \theta_3 \pm \theta_5$, and solving for it uses elements (3,1) and (3,2) of (8) and (9). This results in the following,

$$\begin{aligned}
&\theta_S^{LEFT} = \operatorname{atan2}(n'_z, -s'_z),\\
&\theta_S^{RIGHT} = \operatorname{atan2}(-n'_z, s'_z),\\
&\theta_5 = \begin{cases} \text{wrapToPi}(\theta_1 - \theta_3 - \theta_S^{LEFT}) & \text{for left arm} \\ \text{wrapToPi}(\theta_S^{RIGHT} - \theta_1 - \theta_3) & \text{for right arm} \end{cases}.
\end{aligned}$$

C. Forward Kinematic Solution for the Leg

As with the arm, the forward kinematics of the leg, ${}^{N}T_F$, are straight forward once the DH parameters are derived. The DH parameters for the right leg are shown in Table III and again (1) is used to find the transformation between adjacent joint coordinate frames. The base frame for the leg is at the waist, and its tranformation to the first hip joint is,

$$ {}^{W}T_{L0} = \begin{bmatrix} 0 & -1 & 1 & \pm l_{L1} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -l_{L2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

use a positive l_{L1} for the left leg and a negative l_{L1} for the right leg. We define the transformation from the hip to the foot as

$$ {}^{L0}T_{L6} = {}^{L0}T_{L1}\, {}^{L1}T_{L2}\, {}^{L2}T_{L3}\, {}^{L3}T_{L4}\, {}^{L4}T_{L5}\, {}^{L5}T_{L6}. \tag{26}$$

Thus, when solving for the forward kinematics ${}^{W}T_F$ must be pre-multiplied by ${}^{W}T_{L0}$ and post-multiplied by ${}^{L6}T_F$. Therefore, the FK is calculated as

$$ {}^{W}T_F = {}^{W}T_{L0}\, {}^{L0}T_{L6}\, {}^{L6}T_F. \tag{27}$$

TABLE III. DH PARAMETERS OF THE LEGS

Right leg DH parameters				
Coord. Frame i	θ_i	α_i	a_i	d_i
1	$\theta_1 + \pi/2$	0	0	0
2	$\theta_2 - \pi/2$	$-\pi/2$	0	0
3	θ_3	0	l_{L3}	0
4	θ_4	0	l_{L4}	0
5	θ_5	$\pi/2$	0	0
6	θ_6	0	l_{L5}	0

D. Inverse Kinematic Solution for the Leg

Similar to the IK for the arm the IK for the leg is solved from the hip to the foot by using the transformation ${}^{L0}T_{L6}$. This is obtained by pre-multiplying ${}^{W}T_F$ by ${}^{L0}T_W$ and post-multiplying by ${}^{F}T_{L6}$. Let us write ${}^{L0}T_{L6}$ for the legs in the same way as for the arms in

$$ {}^{L0}T_{L6} = {}^{L0}T_W\, {}^{W}T_F\, {}^{F}T_{L6}. \tag{28}$$

As with the arm, the three hip joint axes in the leg on Hubo intersect at a single point, therefore a closed-form kinematic solution exists. The six tranformation matrices are obtained by plugging the DH parameters into (1) and pre-multiplying them to obtain the position and orientation of the foot relavtive to the hip,

$$ {}^{L0}T_{L6} = \begin{bmatrix} \mathrm{x}_6 & \mathrm{y}_6 & \mathrm{z}_6 & \mathrm{p}_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{n} & \mathrm{s} & \mathrm{a} & \mathrm{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

where x_6, y_6 and z_6 are the unit vectors along the principal axes of the foot frame and describe the orientation of the foot coordinate frame relative the the hip coordinate frame, and p_6 is the position vector describing the location of the foot relative to the hip. The vectors in [n s a p] again represent the normal vector, sliding vector, approach vector and position vector of the foot, respectively [3].

The leg can be viewed in reverse so that the last three joints make up the hip, and now the position and orientation of the hip frame can be described relative to the foot frame. This new position vector, p', is only a function of θ_4, θ_5 and θ_6, and thus decouples the leg into position and orientation components. The IK problem is solved in this reverse method by taking the inverse of both sides of (29),

$$ {}^{L0}T_{L6}{}' = \begin{bmatrix} \mathrm{n} & \mathrm{s} & \mathrm{a} & \mathrm{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}' = \begin{bmatrix} \mathrm{n}' & \mathrm{s}' & \mathrm{a}' & \mathrm{p}' \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{L6}T_{L0}. \tag{30}$$

We can solve for the last three joint angles by equating the position terms of both sides of (30),

$$ -C_6(l_{L3}C_{45} + l_{L4}C_5) = p'_x + l_{L5}, \tag{31}$$
$$ S_6(l_{L3}C_{45} + l_{L4}C_5) = p'_y, \tag{32}$$
$$ -l_{L3}S_{45} - l_{L4}S_5 = p'_z. \tag{33}$$

We have three equations with the three unknowns θ_4, θ_5 and θ_6. We can solve for C_4 by squaring and adding all three equations, and then obtain S_4 from C_4. This results in,

$$C_4 = \frac{(p'_x + l_{L5})^2 + {p'_y}^2 + {p'_z}^2 - l_{L3}^2 - l_{L4}^3}{2 l_{L3} l_{L4}}$$
$$\theta_4 = \mathrm{atan2}\left(\pm\mathrm{real}\left(\sqrt{1 - C_4^2}\right), C_4\right). \quad (34)$$

We can get θ_5 by squaring equations (31) and (32) and adding them to get,

$$C_5(C_4 l_{L3} + l_{L4}) - S_5(S_4 l_{L3}) = \pm\mathrm{real}\left(\sqrt{(p'_x + l_{L5})^2 + {p'_y}^2}\right). \quad (35)$$

Then, we can expand (33) to yield,

$$S_5(C_4 l_{L3} + l_{L4}) + C_5(S_4 l_{L3}) = -p'_z. \quad (36)$$

If we let $C_4 l_{L3} + l_{L4} = rC_\psi$ and $S_4 l_{L3} = rS_\psi$ and substitute these into (35) and (36) we get,

$$rC_{5\psi} = \pm\sqrt{(p'_x + l_{L5})^2 + {p'_y}^2} \quad (37)$$
$$rS_{5\psi} = -p'_z \quad (38)$$

where

$$r = \pm\mathrm{real}\left(\sqrt{(p'_x + l_{L5})^2 + {p'_y}^2 + {p'_z}^2}\right),$$
$$\psi = \mathrm{atan2}(S_4 l_{L3}, C_4 l_{L3} + l_{L4}).$$

We can obtain $\tan(\theta_5 + \psi)$ by dividing (38) by (37), and then taking the inverse. By then subtracting ψ we can solve for the joint angle θ_5,

$$\theta_5 = \mathrm{wrapToPi}(\mathrm{atan2}(-p'_z, \pm\,\mathrm{real}(\sqrt{(p'_x + l_{L5})^2 + {p'_y}^2})) - \psi). \quad (39)$$

To get the joint solution for θ_6 we can divide (32) by (31),

$$\theta_6 = \mathrm{atan2}(p'_y, -p'_x - l_{L5}). \quad (40)$$

If $C_{45} l_{L3} + C_5 l_{L4} < 0$, then $\theta_6 = \mathrm{wrapToPi}(\theta_6 + \pi)$. Just as with the right arm, we will use the reverse method to solve for the joint angles of the right leg. To solve for the last three joint angles (i.e., the knee, ankle roll and ankle pitch joint angles) we can use the inverse transform method, just as we did for the right arm, by multiplying both sides of (30) by ${}^{L5}T_{L6}$. This results in an equation we will call G_5, where the left hand side of the equation is,

$$G_{5-leg}^{(left)} = {}^{L5}T_{L6}\begin{bmatrix} \mathrm{n}' & \mathrm{s}' & \mathrm{a}' & \mathrm{p}' \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} C_6 n'_x - S_6 n'_y, & C_6 s'_x - S_6 s'_y, & C_6 a'_x - S_6 a'_y, & C_6(p'_x + l_{L5}) - S_6 p'_y \\ S_6 n'_x + C_6 n'_y, & S_6 s'_x + C_6 s'_y, & S_6 a'_x + C_6 a'_y, & S_6(p'_x + l_{L5}) + C_6 p'_y \\ n'_z, & s'_z, & a'_z, & p'_z \\ 0, & 0, & 0, & 1 \end{bmatrix} \quad (41)$$

and the right hand side of G_5 is,

$$G_{5-leg}^{(right)} = {}^{L5}T_{L0} = {}^{L5}T_{L4}\,{}^{L4}T_{L3}\,{}^{L3}T_{L2}\,{}^{L2}T_{L1}\,{}^{L1}T_{L0} =$$
$$\begin{bmatrix} g_{511} & g_{512} & g_{513} & -l_{L3}C_{45} - l_{L4}C_5 \\ g_{521} & g_{522} & g_{523} & 0 \\ g_{531} & g_{532} & g_{533} & -l_{L3}S_{45} - l_{L4}S_5 \\ g_{541} & g_{542} & g_{543} & 1 \end{bmatrix}. \quad (42)$$

We can get the joint solution for θ_2 by setting element (2,3) of (41) and (42) equal to get,

$$S_2 = S_6 a'_x + C_6 a'_y \quad (43)$$
$$\theta_2 = \mathrm{atan2}(S_6 a'_x + C_6 a'_y, \pm\mathrm{real}(\sqrt{1 - (S_6 a'_x + C_6 a'_y)^2})). \quad (44)$$

We get the joint solution for θ_1 by comparing elements (2,1) and (2,2) of (41) and (42) and then dividing them to get,

$$C_2 S_1 = S_6 s'_x + C_6 s'_y \quad (45)$$
$$C_2 C_1 = S_6 n'_x + C_6 n'_y \quad (46)$$
$$\theta_1 = \mathrm{atan2}(S_6 s'_x + C_6 s'_y, S_6 n'_x + C_6 n'_y). \quad (47)$$

If $C2 < 0$, then $\theta_1 = \mathrm{wrapToPi}(\theta_1 + \pi)$. Finally, to obtain the joint solution for θ_3 we start by comparing elements (1,3) and (3,3) of (41) and (42) to get two equations, and then divide them to get θ_{345},

$$-C_2 S_{345} = a'_z \quad (48)$$
$$-C_2 C_{345} = C_6 a'_x - S_6 a'_y \quad (49)$$
$$\theta_{345} = \mathrm{atan2}(a'_z, C_6 a'_x - S_6 a'_y) \quad (50)$$

Then from (50) we can obtain θ_3 by subtracting θ_4 and θ_5,

$$\theta_3 = \mathrm{wrapToPi}(\theta_{345} - \theta_4 - \theta_5). \quad (51)$$

As with the arm there are two solutions for θ_2, θ_4, and θ_5, which generate eight total solutions to the leg IK. Lik with the arm, if the goal position is outside the feasible workspace of the limb the joint solutions will have imaginary parts and only the real parts are used.

E. Choosing Joint Solution

For the inverse kinematics of each of the arms and the legs there are eight joint solutions. The sum of squared joint values is the primary metric that is used in picking one of the eight solutions. Choosing the solution that minimizes this metric is the solution that is "closest" to the zero position of the joints. This works well if at least one of the solutions has all of its joints values within the joint limits (Table IV).

If none of the solutions have all the joint values within the limits then there is no solution that satisfies the desired pose (orientation and position). To get the end-effector to a position as close as possible to the desired position the joint values in all the solutions are capped at the closest joint limit value. Each of the solutions are then given to the FK to calculate the end-effector location with the capped joint values. The solution that gets the end-effector position the closest to the desired position is used. If none of the joint solutions get the end-effector within 5 cm of the desired position then the previous joint values are used.

TABLE IV. JOINT LIMITS OF THE ARMS AND LEGS

	Arms				Legs			
Joint	Left		Right		Left		Right	
i	min.	max.	min.	max.	min.	max.	min.	max.
0	-2.0	2.0	-2.0	2.0	0	1.8	-1.8	0
1	-0.3	2.0	-2.0	0.3	0	0.6	-0.6	0
2	-2.0	2.0	-2	2.0	-1.3	1.4	-1.3	1.4
3	-2.5	0	-2.5	0	0	2.5	0	2.5
4	-2.5	2.0	-2.5	2	-1.3	1.8	-1.3	1.8
5	-1.4	1.2	-1.4	1.2	-0.3	0.2	-0.2	0.3

F. Forward and Inverse Kinematics for the Torso

The kinematics for the torso are almost trivial because the torso only has one joint at the waist, but is still worth presenting. The transformation from the waist to the neck is

$$ {}^{W}T_{N} = \begin{bmatrix} C_{\theta_T} & -S_{\theta_T} & 0 & 0 \\ S_{\theta_T} & C_{\theta_T} & 0 & 0 \\ 0 & 0 & 1 & l_T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (52) $$

where θ_T is the angle of the waist joint. Thus, (52) is the FK for the torso. The IK is also just as simple. Given ${}^{W}T_{N}$ in the form

$$ {}^{W}T_{N} = \begin{bmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (53) $$

then the angle of the waist joint is

$$ \theta_T = \operatorname{atan2}(n_y, n_x). \quad (54) $$

G. Forward and Inverse Kinematics for the Head

The kinematics for the head have two joints, so they are slightly more complex than the torso, but still simple. The transformation from the neck to the head is

$$ {}^{N}T_{H} = {}^{N}T_{H0}\ {}^{H0}T_{H1}\ {}^{H1}T_{H2}\ {}^{H2}T_{H} = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & l_{H2}C_1S_2 \\ C_2S_1 & C_1 & S_1S_2 & l_{H2}S_1S_2 \\ -S_2 & 0 & C_1 & l_{H1} + l_{H2}C_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (55) $$

Thus, (55) is the FK for the head given the two joint angles. Given ${}^{N}T_{H}$ in the form

$$ {}^{N}T_{H} = \begin{bmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (56) $$

the IK for the head is solved by

$$ \begin{aligned} \theta_1 &= \operatorname{atan2}(s_y, -s_x), \\ \theta_2 &= \operatorname{atan2}(n_x, a_x). \end{aligned} \quad (57) $$

H. Forward and Inverse Kinematics for the Full Body

With a humanoid robot most tasks are going to be preformed with the hands or with some tool that the hands are manipulating. Therefore, ultimately one would like to be able to specify some pose in the global frame for the end-effector of the hand and have all the joints on the robot move so that pose is reached. To accomplish this full body IK we break the problem up into parts. First, we assume the robot can walk to a location such that the desired hand location is reachable the hand. Next, the legs will be used to position the neck at the height of the desired hand position. This is because the workspace plane for the arms is maximized at shoulder height, which is the same level as the neck frame. Next, the waist will rotate so that the shoulder is as close as possible to the desired hand location. Finally, the arm will move the hand to the desired pose.

Given a desired transformation from the global frame to the end-effector in the form

$$ {}^{G}T_{E} = \begin{bmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (58) $$

where we assume that the robot has already walked to a suitable position, then the IK is as follows:

1) Set ${}^{W}T_{F}$ to identity with the exception of setting the (2,4) element to $p_y - l_T$ and use the leg IK with this transformation to put the neck at the correct height.
2) Calculate ${}^{W}T_{E}$ using the FK of the legs.
3) Set θ_T to $-\operatorname{atan2}({}^{W}p_x, {}^{W}p_y)$ (left hand) or $\pi - \operatorname{atan2}({}^{W}p_x, {}^{W}p_y)$ (right hand), where ${}^{W}p$ is the position vector of ${}^{W}T_{E}$.
4) Calculate ${}^{N}T_{E}$ using the FK of the legs and waist and use that with the IK of the arms to position the end-effector.

III. Conclusion

In this paper we have derived the forward and inverse kinematics for a humanoid robot with 27 degrees of freedom. The specific humanoid robot that this was designed for is the HUBO2+ platform, but this can easily be used for any robot with the same joint configuration. The kinematics solutions were divided up into four parts: arm, leg, torso, and head. Analytical solutions were derived for each part. Then, using these analytical solutions an algorithm is developed to obtain the forward and inverse kinematics for the entire robot.

References

[1] M. A. Ali, H. A. Park, and C. S. G. Lee, "Closed-form Inverse Kinematic Joint Solution for Humanoid Robots," pp. 704–709, 2010.

[2] D. L. Peiper, "The Kinematics of Manipulators Under Computer Control,," Oct. 1968.

[3] K. S. Fu, R. C. Gonzalez, and C. S. G. Lee, *Robotics: Control, Sensing, Vision, and Intelligence*. McGraw-Hill, Inc., Jan. 1987.